Real-time thermal Schwinger-Dyson equation for quark self-energy in Landau gauge *

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By means of a formal expression of the Cornwall-Jackiw-Tomboulis effective potential for quark propagator at finite temperature and finite quark chemical potential, we derive the real-time thermal Schwinger-Dyson equation for quark propagator in Landau gauge. Denote the inverse quark propagator by $A(p^2)\not{p} - B(p^2)$, we argue that, when temperature T is less than the given infrared momentum cutoff p_c, $A(p^2) = 1$ is a feasible approximation and can be assumed in discussions of chiral symmetry phase transition problem in QCD.

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I. INTRODUCTION

The Schwinger-Dyson (SD) equation for quark propagator is one of strong tools to research dynamical chiral symmetry breaking. A great deal of work on it in the zero temperature case has been made and a full demonstration of chiral symmetry breaking in zero temperature Quantum Chromadynamics (QCD) has been given [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. However, the Schwinger-Dyson approach of chiral symmetry at finite temperature has not been fully explored. Such research is of important significance for understanding of the chiral phase transition at high temperature and even the color superconducting phase transition at low temperature and high density in QCD [11], since the SD approach is a nonperturbative one which will give more reliable results.

In present work on the SD approach of chiral phase transition, most are based on the imaginary-time formalism of thermal field theory [12] and very few were seen in the literature which are based on the real-time formalism. In fact, in the real-time formalism, one can directly use some results in the zero temperature QCD consistent with the renormalization group (RG) analysis, thermalize them by thermal transformation matrices and obtain the expected expressions at finite temperature and chemical potential. In particular, the research in zero temperature case has shown that in the Landau gauge, the calculations can be simplified greatly [9]. It is expected that at a finite temperature, this advantage will be maintained if the Landau gauge is still taken.

The SD equation for quark propagator can be directly derived from the extreme value condition of corresponding Cornwall-Jackiw-Tomboulis (CJT) effective potential [13]. In fact, if being limited to derive the SD equation, one needs only a formal expression for the effective potential rather than its a final explicit one. Conversely, the discussions based the SD equation of the mass function in quark propagator will be quite useful to simplifying the explicit calculation of corresponding CJT effective potential for quark propagator and this is also one of purposes of our research in this paper.

The paper is arranged as follows. In Sect.II we will give a formal expression of CJT effective potential for quark propagator at finite temperature and finite quark chemical potential in the real-time formalism of thermal field theory and from which in Sect.III, we will derive in Landau gauge the SD equation for quark self-energy in finite temperature and finite chemical potential case. In Sect.IV an important possible approximation for the quark propagator will be explored and its practical significance for chiral phase transition problem in QCD will be indicated.

II. THE THERMAL CJT EFFECTIVE POTENTIAL FOR QUARK PROPAGATOR

It is well known that at temperature $T = 0$ and the quark chemical potential $\mu = 0$, the global flavor chiral symmetries in QCD will be spontaneously broken owing to the vacuum quark-antiquark condensates $\langle\bar{\psi}\psi\rangle \neq 0$ [8, 9, 14, 15], this makes quarks acquire their dynamical masses. For research on chiral symmetry restoring which could occur at a finite T and μ, the thermal effective potential for quark propagator is essential. We will derive a formal expression of the effective potential first by writing the CJT effective action $\Gamma[G, G^*]$ in QCD for the quark propagator G and its conjugate G^* at finite T and μ by

$$\Gamma_T[G, G^*] = \Gamma_1[G, G^*] + \Gamma_2[G, G^*], \tag{1}$$

where $\Gamma_1[G, G^*]$ and $\Gamma_2[G, G^*]$ are respectively the contribution from one-loop vacuum diagram and two and more loop vacuum diagrams without quark self-energy

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correction, since G in Eq.(1) is the exact quark propagator. In the real-time formalism of thermal field theory, we have

$$\begin{aligned}\Gamma_1[G,G^*] &= -i\mathrm{Tr}(\ln S_T G_T^{-1})^{11} \\ &\quad -i\mathrm{Tr}(S_T^{-1}G_T)^{11} + i\mathrm{Tr}1,\end{aligned} \tag{2}$$

where S_T and G_T are 2×2 thermal matrix propagators, the superscript "11" represents the 11 component of the corresponding matrix and the Tr is in functional sense. Assuming translational invariance, then in momentum space, we have [16]

$$S_T(p) = M_p\tilde{S}(p)M_p, \;\; G_T(p) = M_p\tilde{G}(p)M_p \tag{3}$$

with the thermal transformation matrix M_p defined by

$$M_p = \begin{pmatrix} \cos\theta_p & -e^{\beta\mu/2}\sin\theta_p \\ e^{\beta\mu/2}\sin\theta_p & \cos\theta_p \end{pmatrix}, \;\; \beta = 1/T$$

$$\sin^2\theta_p = \theta(p^0)\tilde{n}(p^0-\mu) + \theta(-p^0)\tilde{n}(-p^0+\mu),$$

$$\tilde{n}(p^0-\mu) = 1/[e^{\beta(p^0-\mu)}+1] \tag{4}$$

and

$$\tilde{S}(p) = \begin{pmatrix} S(p) & 0 \\ 0 & S^*(p) \end{pmatrix}, \;\; \tilde{G}(p) = \begin{pmatrix} G(p) & 0 \\ 0 & G^*(p) \end{pmatrix} \tag{5}$$

where

$$S(p) = i/(\not{p}+i\varepsilon), \;\; S^*(p) = -i/(\not{p}-i\varepsilon), \;\; \not{p} = \gamma^\mu p_\mu \tag{6}$$

and

$$\begin{aligned} G(p) &= i/[A(p^2)\not{p} - B(p^2) + i\varepsilon], \\ G^*(p) &= -i/[A(p^2)\not{p} - B(p^2) - i\varepsilon]. \end{aligned} \tag{7}$$

$G(p)$ and $G^*(p)$ are the complete propagators of quarks with dynamical mass, where $A(p^2)$ and $B(p^2)$ have been assumed to be real functions. Thus in the momentum space,

$$\begin{aligned}\Gamma_1[G,G^*] &= -\Omega i N_f N_c \left\{\left\langle \mathrm{tr}[\ln S_T(p)G_T^{-1}(p)]^{11}\right\rangle \right. \\ &\quad \left. + \left\langle \mathrm{tr}[S_T^{-1}G_T(p)]^{11}\right\rangle - \left\langle \mathrm{tr}1\right\rangle\right\} \\ &\equiv -\Omega V_1[G,G^*]\end{aligned} \tag{8}$$

where N_f and N_c are respectively the number of flavor and color of the quarks, Ω is the volume of space-time, tr only represents the trace of spinor matrices and the denotation $\langle\cdots\rangle$ means the integration $\int d^4p/(2\pi)^4$. By means of Eqs.(3)-(6) and the relation [17]

$$\ln S_T(p)G_T^{-1}(p) = M_p \ln[\tilde{S}(p)\tilde{G}^{-1}(p)]M^{-1}(p),$$

we obtain from Eq.(8) the one-loop effective potential $V_1[G,G^*]$ corresponding to $\Gamma_1[G,G^*]$

$$\begin{aligned} V_1[G,G^*] &= iN_fN_c\left(\left\langle \cos^2\theta_p\left\{\mathrm{tr}\ln[S(p)G^{-1}(p)] + \mathrm{tr}[S^{-1}(p)G(p)]\right\}\right\rangle\right. \\ &\quad \left. + \left\langle \sin^2\theta_p\left\{\mathrm{tr}\ln[S^*(p)G^{*-1}(p)] + \mathrm{tr}[S^{*-1}(p)G^*(p)]\right\}\right\rangle - \left\langle \mathrm{tr}1\right\rangle\right). \end{aligned} \tag{9}$$

In the real-time formalism of thermal field theory, the interaction Lagrangian between the quark fields ψ and the gluon fields $A_\mu^a (a = 1,\cdots,8)$ can be expressed by

$$\mathcal{L}_i = -\sum_{a=1}^{8}\sum_{r=1,2} g_0(-1)^{r+1}\bar{\psi}^{(r)}\gamma^\mu A_\mu^{a(r)}\psi^{(r)}\lambda^a(\underline{N}_c),$$

where $r=1$ and $r=2$ correspond to physical and (thermal) ghost fields respectively, $\lambda^a(\underline{N}_c)$ is the generator matrix of the color gauge group $SU_c(3)$ in the representation $\underline{N}_c$ of the quarks and g_0 is the bare gauge coupling constant. Assume that only the two-loop vacuum diagrams are included in $\Gamma_2[G,G^*]$, then we can write in the momentum space

$$\begin{aligned}\Gamma_2[G,G^*] &= -\frac{i}{2}\left\langle\left\langle g_0^2\mathrm{Tr}\gamma^\mu\lambda^a(\underline{N}_c)G_T^{11}(p)\gamma^\nu\lambda^b(\underline{N}_c)G_T^{11}(q)[D'^{ab}_{\mu\nu}(p-q)]_T^{11}\right\rangle\right\rangle N_f\Omega \\ &\quad + \frac{i}{2}\left\langle\left\langle g_0^2\mathrm{Tr}\gamma^\mu\lambda^a(\underline{N}_c)G_T^{12}(p)\gamma^\nu\lambda^b(\underline{N}_c)G_T^{21}(q)[D'^{ab}_{\mu\nu}(p-q)]_T^{12}\right\rangle\right\rangle N_f\Omega \\ &\equiv -\Omega V_2[G,G^*]\end{aligned} \tag{10}$$

where $[D'^{ab}_{\mu\nu}(p-q)]_T$ is the complete thermal gluon matrix propagator, $\langle\langle\cdots\rangle\rangle$ represents the integrations $\int\int d^4pd^4q/(2\pi)^8$ and Tr is now only for the quark-dependent matrices (flavor, color, and spinor etc.). After

the renormalization procedure is implemented, we can replace g_0^2 by the RG-invariant running gauge coupling $g^2\big[(p-q)^2\big]$, and $\big[D'^{ab}_{\mu\nu}(p-q)\big]_T$ by the tree diagram gluon propagator $\big[D^{ab}_{\mu\nu}(p-q)\big]_T$, i.e. we will take the approximation [8, 9]

$$g_0^2\big[D'^{ab}_{\mu\nu}(p-q)\big]_T \simeq g^2\big[(p-q)^2\big]\big[D^{ab}_{\mu\nu}(p-q)\big]_T. \quad (11)$$

In Eq.(10) the complete thermal vertices $\Gamma^{\mu(r)}(p,q)(r=1,2)$ have been replaced by the tree diagram vertex γ^μ. This is because when $A(p^2)$ and $B(p^2)$ in the inverse quark propagator $G^{-1}(p)$ are real functions, the Ward-Takahashi (WT) identity at finite T and μ (if the ghost effect in the fermion sector is neglected) essentially identical to the one at $T=\mu=0$, thus in the Landau gauge we can make the couplings between the complete vertices $\Gamma^{\mu(r)}(p,q)(r=1,2)$ which submit to the WT identity and the thermal gluon propagator are equivalent to the one between the tree diagram vertex γ^μ and the thermal gluon propagator [18]. The tree diagram thermal gluon matrix propagator $[D^{ab}_{\mu\nu}(k)]_T$ can be expressed by [16]

$$\big[D^{ab}_{\mu\nu}(k)\big]_T = \big[D_{\mu\nu}(k)\big]_T\delta^{ab},\quad \big[D_{\mu\nu}(k)\big]_T = \bar{M}_k\tilde{D}_{\mu\nu}(k)\bar{M}_k \quad (12)$$

with the thermal transformation matrix $\bar{M}_k$ defined by

$$\bar{M}_k = \begin{pmatrix} \cosh\Theta_k & \sinh\Theta_k \\ \sinh\Theta_k & \cosh\Theta_k \end{pmatrix},\quad \sinh\Theta_k=\sqrt{n(k^0)},$$
$$n(k^0)=\frac{1}{e^{\beta|k^0|}-1} \quad (13)$$

and

$$\tilde{D}_{\mu\nu}(k) = \begin{pmatrix} D_{\mu\nu}(k) & 0 \\ 0 & D^*_{\mu\nu}(k) \end{pmatrix}. \quad (14)$$

In Landau gauge,

$$\left\{\begin{array}{l} D_{\mu\nu}(k) \\ D^*_{\mu\nu}(k) \end{array}\right. = \frac{\mp i}{k^2\pm i\varepsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2\pm i\varepsilon}\right). \quad (15)$$

By means of Eqs.(3)-(5) and Eqs. (11)-(14), we may obtain from Eq.(10) the two-loop effective potential $V_2[G,G^*]$ corresponding to $\Gamma_2[G,G^*]$, expressed by

$$\begin{aligned} V_2[G,G^*] &= \frac{i}{2}N_fN_cC_2(\underline{N}_c)\big\langle\big\langle g^2\big[(p-q)^2\big]\mathrm{tr}\big[\gamma^\mu G^{11}_T(p)\gamma^\nu G^{11}_T(q)\big]\big[D_{\mu\nu}(p-q)\big]^{11}_T\big\rangle\big\rangle \\ &\quad -\frac{i}{2}N_fN_cC_2(\underline{N}_c)\big\langle\big\langle g^2\big[(p-q)^2\big]\mathrm{tr}\big[\gamma^\mu G^{12}_T(p)\gamma^\nu G^{21}_T(q)\big]\big[D_{\mu\nu}(p-q)\big]^{12}_T\big\rangle\big\rangle. \end{aligned} \quad (16)$$

with

$$G^{11}_T(p)=\cos^2\theta_pG(p)-\sin^2\theta_pG^*(p),\quad G^{12}_T(p)=-\cos\theta_p\,\sin\theta_p\,e^{\beta\mu/2}[G(p)+G^*(p)]=-e^{\beta\mu}G^{21}_T(p), \quad (17)$$

$$\big[D_{\mu\nu}(p-q)\big]^{11}_T = D_{\mu\nu}(p-q)-g_{\mu\nu}2\pi n(p^0-q^0)\delta\big[(p-q)^2\big],\quad \big[D_{\mu\nu}(p-q)\big]^{12}_T = -g_{\mu\nu}e^{\beta|p^0-q^0|/2}2\pi n(p^0-q^0)\delta\big[(p-q)^2\big] \quad (18)$$

and

$$N_cC_2(\underline{N}_c)=\sum_{a,b}\mathrm{tr}\big[\lambda^a(\underline{N}_c)\lambda^b(\underline{N}_c)\big],\quad C_2(\underline{N}_c)=\frac{N_c^2-1}{2N_c}.$$

The total effective potential will be

$$V[G,G^*]=V_1[G,G^*]+V_2[G,G^*] \quad (19)$$

III. THE THERMAL SCHWINGER-DYSON EQUATION FOR QUARK SELF-ENERGY

The effective potential $V[G,G^*]$ is in fact the functional of the thermal propagator $\big[G(p)\big]^{rs}_T$ and $\big[D^{ab}_{\mu\nu}(p-q)\big]^{rs}_T$ $(rs=11,12)$. The Schwinger-Dyson equation for quark self-energy will come from the extremum condition

$$\frac{\delta V[G,G^*]}{\delta G^{11}_T(p)}=0. \quad (20)$$

In view of Eq.(17), equation (20) is equivalent to

$$\left[\frac{\delta}{\delta G(p)}-\frac{\delta}{\delta G^*(p)}\right]V[G,G^*]=0$$

which, by using Eqs.(9), (16) and (19), can be reduced to

$$\begin{aligned}\Sigma(p) &\equiv [1-A(p^2)]\,\not{p}+B(p^2)\\ &= -iC_2(\underline{N}_c)\big\langle g^2[(p-q)^2]\gamma^\mu G_T^{11}(q)\gamma^\nu[D_{\mu\nu}(p-q)]_T^{11}\big\rangle\end{aligned} \tag{21}$$

Equation (21) is just the SD equation of the complete self-energy $\Sigma(p)$ for single flavor and single color quark to one-loop order at finite T and μ. It is noted that the second term in the right-handed side of Eq.(16) has no contribution to Eq.(21). By acting tr$\not{p}$ and tr on the two sides of Eq.(21) separately, we can obtain the equations of $A(p^2)$ and $B(p^2)$,

$$A(p^2)=1+i\frac{C_2(\underline{N}_c)}{4p^2}\Big\langle g^2\big[(p-q)^2\big]\mathrm{tr}\big[\not{p}\gamma^\mu G_T^{11}(q)\gamma^\nu\big]\big[D_{\mu\nu}(p-q)\big]_T^{11}\Big\rangle \tag{22}$$

$$B(p^2)=-i\frac{C_2(\underline{N}_c)}{4}\Big\langle g^2\big[(p-q)^2\big]\mathrm{tr}\big[\gamma^\mu G_T^{11}(q)\gamma^\nu\big]\big[D_{\mu\nu}(p-q)\big]_T^{11}\Big\rangle. \tag{23}$$

From Eqs.(7) and (17) we have

$$G_T^{11}(q)=\big[A(q^2)\,\not{q}+B(q^2)\big]\left\{\frac{i}{A^2(q^2)q^2-B^2(q^2)+i\varepsilon}-2\pi\sin^2\theta_q\delta\big[A^2(q^2)q^2-B^2(q^2)\big]\right\}. \tag{24}$$

By means of Eqs.(18) and (24), and the result that

$$\mathrm{tr}\big(\not{p}\gamma^\mu\,\not{q}\gamma^\nu\big)D_{\mu\nu}(p-q)=-i4E(p,q)$$

with

$$E(p,q)=1-\frac{1}{2}\left[p^2+q^2+\frac{(p^2-q^2)^2}{(p-q)^2}\right]/(p-q)^2, \tag{25}$$

equation (22) can be changed into

$$\begin{aligned}A(p^2) &= 1+i\frac{C_2(\underline{N}_c)}{p^2}\Big\langle g^2\big[(p-q)^2\big]A(q^2)\Big\{\frac{i}{A^2(q^2)q^2-B^2(q^2)+i\varepsilon}-2\pi\sin^2\theta_q\delta\big[A^2(q^2)q^2-B^2(q^2)\big]\Big\}\\ &\quad\times\Big\{-iE(p,q)+4\pi(p\cdot q)n(p^0-q^0)\delta\big[(p-q)^2\big]\Big\}\Big\rangle.\end{aligned} \tag{26}$$

It is obtained that the integration in Eq.(26)

$$\Big\langle g^2\big[(p-q)^2\big]A(q^2)\frac{E(p,q)}{A^2(q^2)q^2-B^2(q^2)+i\varepsilon}\Big\rangle=-i\int\frac{d^4\bar{q}}{(2\pi)^4}g^2[(\bar{p}-\bar{q})^2]A(\bar{q}^2)\frac{E(\bar{p},\bar{q})}{A^2(\bar{q}^2)\bar{q}^2+B^2(\bar{q}^2)}=0,$$

where we have made the changes of variables $\bar{p}^0=ip^0,\ \bar{p}^i=p^i\ (i=1,2,3)$ and the same to $q\to\bar{q}$ after the Wick rotation, and the assumption that [9]

$$g^2\big[(\bar{p}-\bar{q})^2\big]=\theta(\bar{p}^2-\bar{q}^2)g^2(\bar{p}^2)+\theta(\bar{q}^2-\bar{p}^2)g^2(\bar{q}^2),$$

and then consider the fact that

$$\int d\Omega_{\bar{q}}E(\bar{p},\bar{q})=0.$$

As a result, equation (26) is reduced to

$$\begin{aligned}A(p^2) &= 1-2\pi\frac{C_2(\underline{N}_c)}{p^2}\Big\langle g^2\big[(p-q)^2\big]A(q^2)\Big\{E(p,q)\sin^2\theta_q\delta\big[A^2(q^2)q^2-B^2(q^2)\big]\\ &\quad+\Big(\frac{1}{A^2(q^2)q^2-B^2(q^2)+i\varepsilon}+i2\pi\sin^2\theta_q\delta\big[A^2(q^2)q^2-B^2(q^2)\big]\Big)\cdot 2(p\cdot q)n(p^0-q^0)\delta\big[(p-q)^2\big]\Big\}\Big\rangle.\end{aligned} \tag{27}$$

On the other hand, by using Eqs. (18) and (24), equation (23) can be changed into

$$B(p^2) = -iC_2(\underline{N}_c)\Big\langle g^2\big[(p-q)^2\big]B(q^2)\Big\{\frac{i}{A^2(q^2)q^2-B^2(q^2)+i\varepsilon}-2\pi\sin^2\theta_q\delta\big[A^2(q^2)q^2-B^2(q^2)\big]\Big\} \times\Big\{\frac{-3i}{(p-q)^2+i\varepsilon}-8\pi n(p^0-q^0)\delta\big[(p-q)^2\big]\Big\}\Big\rangle. \tag{28}$$

Equations (27) and (28) are the required coupled equations of the function $A(p^2)$ and $B(p^2)$ in the quark self-energy $\Sigma(p)$ in Landau gauge in the real-time thermal QCD. It is indicated that when $T=\mu=0$, all the thermal correction terms can be removed, hence equations (27) and (28) will be reduced to

$$A(p^2)=1,\;\; B(p^2)=-i3C_2(\underline{N}_c)\Big\langle g^2\big[(p-q)^2\big]B(q^2)\frac{1}{q^2-B^2(q^2)+i\varepsilon}\cdot\frac{1}{(p-q)^2+i\varepsilon}\Big\rangle \tag{29}$$

which are just the SD equations for quark self-energy function in Landau gauge at $T=\mu=0$.

IV. DISCUSSION OF $A(p^2)$

Equation (27) shows that, at a finite T and μ, $A(p^2)$ has extra thermal correction terms to unity. Based on the consideration of dimension, $A(p^2)$ can be generally written by $A(p^2)=1-(T^2/p^2)F$, where F depends on only the dimensionless variants consisting of T, p, B,μ etc.. Assuming F is finite, then it can be deduced that at a high T^2 and a low p^2, the thermal correction terms would be not small and $A(p^2)=1$ would not be a good approximation. However, we will argue that for some physically interesting range of T, the thermal corrections of $A(p^2)$ are negligible hence the approximation $A(p^2)=1$ may still be valid.

Denote the quark mass function by

$$m^2(q^2)=B^2(q^2)/A^2(q^2), \tag{30}$$

then $A(p^2)$ can be expressed by

$$A(p^2) = 1-\frac{C_2(\underline{N}_c)}{p^2}\int\frac{d^4q}{(2\pi)^3}\frac{1}{A(q^2)}\Big\{g^2\big[(p-q)^2\big]E(p,q)\sin^2\theta_q\delta\big[q^2-m^2(q^2)\big] +g^2(0)\Big(\frac{1}{q^2-m^2(q^2)+i\varepsilon}+i2\pi\sin^2\theta_q\delta\big[q^2-m^2(q^2)\big]\Big)(p^2+q^2)n(p^0-q^0)\delta\big[(p-q)^2\big]\Big\}. \tag{31}$$

Let $q^2=m_1^2$ is a real root of the equation $q^2=m^2(q^2)$, i.e. m_1 obeys the equation

$$m_1^2=m^2(m_1^2), \tag{32}$$

then we will have

$$\delta\big[q^2-m^2(q^2)\big]=\frac{1}{f(m_1^2)}\delta\big[q^2-m_1^2\big)],\;\; f(m_1^2)=\Big|1-\frac{\partial m^2(q^2)}{\partial q^2}\Big|_{q^2=m_1^2}. \tag{33}$$

In fact, one can identify $m(q^2)$ with the running quark mass function consistent with the renormalization group (RG) in QCD [8, 9], then it will be a descent function of q^2 and we always have $f(m_1^2)\geq 1$. To estimate the order of magnitude of the thermal corrections of $A(p^2)$, we will take the approximation

$$g^2\big[(p-q)^2\big]E(p,q) \simeq \theta(|p^2|-|q^2|)\,g^2(p^2)\left\{1-\frac{1}{2}\left[p^2+q^2+\frac{(p^2-q^2)^2}{p^2}\right]\frac{1}{p^2}\right\} +\theta(|q^2|-|p^2|)\,g^2(q^2)\left\{1-\frac{1}{2}\left[p^2+q^2+\frac{(p^2-q^2)^2}{q^2}\right]\frac{1}{q^2}\right\}$$

and

$$\frac{1}{A(q^2)}\cdot\frac{1}{q^2-m^2(q^2)+i\varepsilon}\simeq\frac{1}{A(m_1^2)f(m_1^2)}\cdot\frac{1}{q^2-m_1^2+i\varepsilon}.$$

Considering the results that

$$\int d^4q \sin^2\theta_q \delta(q^2-m_1^2) = 4\pi T^2 I_3(y_1,r)$$

with

$$I_3(y_1,r) = \frac{1}{2}\int_0^\infty dx \frac{x^2}{\sqrt{x^2+y_1^2}}\left[\frac{1}{e^{\sqrt{x^2+y_1^2}-r}+1}+(-r\to r)\right],\ y_1=\frac{m_1}{T},\ r=\frac{\mu}{T}$$

and

$$\int d^4q n(q^0)\delta(q^2) = \frac{2}{3}\pi^3T^2,$$

we can write

$$A(p^2) = 1 - a(p^2)/A(m_1^2), \tag{34}$$

where

$$\begin{aligned} a(p^2) \;=\; & \frac{1}{f(m_1^2)}\left\{\left[\frac{2}{3\pi^2}\tilde{g}^2(p^2)I_3(y_1,r)+\frac{g^2(0)}{9}\right]\frac{T^2}{(|p|^2+p_c^2)\varepsilon(p^2)}\right. \\ & \left.+\frac{g^2(0)}{12\pi^2}\frac{p^2+m_1^2}{(|p|^2+p_c^2)\varepsilon(p^2)}\int_0^\infty\frac{dx}{e^x-1}\left[\ln\frac{p^2-m_1^2+2xT(p^0+|\vec{p}|)}{p^2-m_1^2+2xT(p^0-|\vec{p}|)}+(p^0\to -p^0)\right]\frac{T}{|\vec{p}|}\right\} \end{aligned} \tag{35}$$

with

$$\tilde{g}^2(p^2) = \theta(m_1^2-|p^2|)g^2(m_1^2)\frac{p^2}{2(m_1^2+p_c^2)}\left(1-\frac{p^2}{m_1^2}\right)+\theta(|p^2|-m_1^2)g^2(p^2)\frac{m_1^2}{2p^2}\left(1-\frac{m_1^2}{p^2}\right). \tag{36}$$

In Eq.(35) we have introduced the infrared momentum cutoff p_c^2 by the replacement $1/p^2 \to 1/(|p|^2+p_c^2)\varepsilon(p^2)$ with $\varepsilon(p^2)\equiv p^2/|p^2|$ and in Eq.(36) by the replacement $1/2m_1^2 \to 1/2(m_1^2+p_c^2)$. When $p^2=m_1^2$, both $\tilde{g}^2(p^2)$ and the integral with the logarithm functions in Eq.(35) are equal to zeroes, hence we get

$$a(m_1^2) = \frac{g^2(0)T^2}{9f\big(m_1^2\big)\big(m_1^2+p_c^2\big)}. \tag{37}$$

When $p^2\neq m_1^2$ and if $|p^2-m_1^2|>T^2$ is assumed, we can expand the logarithm functions in Eq.(35) in power of $2xT(p^0\pm|\vec{p}|)/(p^2-m_1^2)$ (noting that the main contribution to the integral over x comes from the region of $x<1$), then in the approximation with the infrared momentum cutoff p_c^2 included that

$$\begin{aligned} \frac{1}{\big(p^2-m_1^2\big)^n} \;=\; & \theta(m_1^2-|p^2|)\frac{1}{\big[-(m_1^2+p_c^2)\big]^n} \\ & +\theta(|p^2|-m_1^2)\frac{1}{\big(p^2\big)^n}, \end{aligned}$$

we can obtain that

$$\begin{aligned} a(p^2) \;=\; & \theta(m_1^2-|p^2|)2a(m_1^2)\frac{|p^2|}{|p^2|+p_c^2}\left\{\frac{3g^2(m_1^2)}{2\pi^2g^2(0)}\left(1-\frac{p^2}{m_1^2}\right)I_3(y_1,r)-1-\frac{4\pi^2}{15}\frac{3{p^0}^2+\vec{p}^2}{p^2}\frac{T^2}{m_1^2+p_c^2}-\cdots\right\} \\ & +\theta(|p^2|-m_1^2)2a(m_1^2)\frac{m_1^2+p_c^2}{(|p|^2+p_c^2)\varepsilon(p^2)}\left\{\frac{3g^2(p^2)}{2\pi^2g^2(0)}\left(1-\frac{m_1^2}{p^2}\right)I_3(y_1,r)\frac{m_1^2}{p^2}+1+\frac{4\pi^2}{15}\frac{3{p^0}^2+\vec{p}^2}{p^2}\frac{T^2}{p^2}+\cdots\right\}. \end{aligned} \tag{38}$$

From Eq.(34) we can get the equation obeyed by $A(m_1^2)$

$$A(m_1^2) = 1 - a(m_1^2)/A(m_1^2)$$

which has the solution

$$A(m_1^2) = \frac{1}{2}\{1+[1-4a(m_1^2)]^{1/2}\}. \tag{39}$$

When temperature T is low enough so that $4a(m_1^2) < 1$, $A(m_1^2)$ will have the order of magnitude of one, and it is seen from Eq.(38) that we always have $a(p^2) \ll 1$, whether $|p^2| \ll m_1^2$ or $|p^2| \gg m_1^2$. Hence we can obtain

$$A(p^2) \simeq 1, \quad \text{for} \;\; \frac{T^2}{m_1^2 + p_c^2} < \frac{9f(m_1^2)}{4g^2(0)}. \tag{40}$$

When temperature T is neither very low nor very high so that $4a(m_1^2) > 1$ but $T^2/(m_1^2 + p_c^2) < 1$, the series expansion in $T^2/(m_1^2 + p_c^2)$ in Eq.(38) may still be used. In this case, $A(m_1^2)$ becomes complex and we may write the module of $a(p^2)/A(m_1^2)$ by

$$\left|\frac{a(p^2)}{A(m_1^2)}\right| = \frac{|a(p^2)|}{|1 + i\sqrt{4a(m_1^2) - 1}|/2} = \frac{|a(p^2)|}{a^{1/2}(m_1^2)}. \tag{41}$$

Noting that the terms contained in the brace in Eq.(38) have the order of magnitude of unity, the module may be approximately estimated by

$$\frac{|a(p^2)|}{a^{1/2}(m_1^2)} \sim \theta(m_1^2 - |p^2|)2a^{1/2}(m_1^2)\frac{|p^2|}{|p^2| + p_c^2} + \theta(|p^2| - m_1^2)2a^{1/2}(m_1^2)\frac{m_1^2 + p_c^2}{(|p|^2 + p_c^2)}. \tag{42}$$

In spite of $2\sqrt{a(m_1^2)} > 1$, when $|p^2| > m_1^2$, $|a(p^2)|/a^{1/2}(m_1^2) \ll 1$ is obvious if p^2 is large enough. On the other hand, when $|p^2| < m_1^2$, we may make a numerical estimation of $|a(p^2)|/a^{1/2}(m_1^2)$ in practical case of QCD. If the running gauge coupling $g^2(p^2)$ in QCD with the infrared momentum cutoff p_c^2 is taken as [8]

$$g^2(p^2) = 2\pi^2 A/\ln\frac{|p^2| + p_c^2}{\Lambda_{QCD}^2},$$

where $A = 24/(33 - 2N_f)$ and Λ_{QCD} is the RG-invariant mass scale parameter, then the quark mass function $m(p^2)$ (when only dynamical quark mass is involved) can be expressed by

$$m^2(p^2) = \Big(\frac{2\pi^2 A}{3}\Big)^2\Big(\frac{\phi}{p^2 + p_c^2}\Big)^2\Big(\ln\frac{|p^2| + p_c^2}{\Lambda_{QCD}^2}\Big)^{A-2},$$

where ϕ is the RG-invariant quark-antiquark condensates [8]. Thus Eqs.(32) and (33) lead to

$$m_1 = \frac{2\pi^2 A}{3}\frac{|\phi|}{m_1^2 + p_c^2}\Big(\ln\frac{m_1^2 + p_c^2}{\Lambda_{QCD}^2}\Big)^{A/2-1}$$

and

$$f(m_1^2) = 1 + \frac{2m_1^2}{m_1^2 + p_c^2}\left[1 + \frac{2 - A}{2}\Big(\ln\frac{m_1^2 + p_c^2}{\Lambda_{QCD}^2}\Big)^{-1}\right].$$

It may be checked that, taking $p^2 = m_1^2/2$, when $N_f = 3$ and $t_c \equiv \ln(p_c^2/\Lambda_{QCD}^2) = 0.1$, for $|\phi|/\Lambda_{QCD}^3 = 0.1$, we will have $m_1/\Lambda_{QCD} \simeq 0.64$ and obtain $|a(p^2)|/a^{1/2}(m_1^2) = 0.5$ and 0.86 respectively for $T/(m_1^2 + p_c^2) = 0.5$ and $T/p_c = 1$; for $|\phi|/\Lambda_{QCD}^3 = 0.5$, we will have $m_1/\Lambda_{QCD} \simeq 1.2$ and obtain $|a(p^2)|/a^{1/2}(m_1^2) = 1.33$ and 1.75 respectively for $T/(m_1^2 + p_c^2) = 0.5$ and $T/p_c = 1$. These results show that when $|p^2| < m_1^2$, $A(p^2) \simeq 1$ is not a good approximation. However, the difference between $A(p^2)$ and 1 is not very large and it is easy to verify that it will decrease as T goes down and the infrared momentum cutoff p_c^2 goes up. It is also seen that the value of m_1^2 either lower or slightly higher than the infrared momentum cutoff p_c^2, so the region in which $|p^2| < m_1^2$ is quite limited, and it may be presumed that the deviation of $A(p^2)$ to 1 will cause only quite small effect to the total results. Hence we can neglect such deviation and take $A(p^2) \simeq 1$ for all the value of p^2 in the conditions

$$\frac{9f(m_1^2)}{4g^2(0)} < \frac{T^2}{m_1^2 + p_c^2} < 1. \tag{43}$$

To sum up, we can assume that

$$A(p^2) \simeq 1, \text{ for } 0 \le T^2 < m_1^2 + p_c^2 \tag{44}$$

is a feasible approximation. When only dynamical quark mass is involved, the minimal m_1 will be zero, and when a current quark mass is also included, the minimal m_1 may still be quite small, so the allowed highest temperature limited by Eq.(44) may be reasonably supposed to be $T = p_c = e^{t_c/2}\Lambda_{QCD} = 1.05\Lambda_{QCD}$ (e.g. for $t_c = 0.1$) and will increases as a larger p_c is taken. Hence, if the chiral phase transition temperature is lower than Λ_{QCD} [12, 19], then the above arguments indicate that within the whole range of temperature related to chiral phase transition, $A(p^2) = 1$ is a feasible approximation and may be used in the discussion of chiral phase transition problem in QCD based on Schwinger-Dyson approach. This will certainly greatly simplify calculations in this nonperturbative method.

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